12/01 Note

Ex: Compute the Flux of $\vec{F} = \langle y, x, z \rangle$ across the boundary of the solid enclosed by paraboloid $z = 1 - x^2 - y^2$ and plane $z = 0$.

Sol: $\iint_S \vec{F}\, ds = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_u)\, dA$

Para S_1: $\vec{r}(u,v) = \langle u\cos(u), u\sin(u), 0 \rangle$
on domain $[0,1] \times [0, 2\pi]$

Para S_2: $\vec{S}(u,v) = \langle u\cos(v), u\sin(v), 1-u^2 \rangle$
on domain $[0,1] \times [0, 2\pi]$

$\Rightarrow$ So: $\iint_S \vec{F} \cdot d\vec{s} = \iint_{D_1} \vec{F} (\vec{r}_u \times \vec{r}_v)\, dA + \iint_{D_2} \vec{F} (\vec{S}_u \times \vec{S}_v)\, dA$

$\Rightarrow$ consider orientation:

$\vec{r}_u = \langle \cos(u), \sin(v), 0 \rangle \quad \vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$$\therefore \vec{r}_u \times \vec{r}_v = \det \begin{vmatrix} i & j & k \\ \cos(u) & \sin(v) & 0 \\ -u\cos(v) & u\cos(v) & 0 \end{vmatrix} = \langle 0, 0, u \rangle$$

$\Rightarrow \vec{S}_u = \langle \cos(v), \sin(v), -2u \rangle \quad \vec{S}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$$\therefore \vec{S}_u \times \vec{S}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(u) & 0 \end{vmatrix} = u\langle 2u\cos(u), 2u\sin(u), 1 \rangle$$

$\Rightarrow \vec{F}$ on S_1 is given by $\vec{F}(\vec{r}(u,v)) = \langle u\sin(v), u\cos(v), 0 \rangle$

$\vec{F}$ on S_2 is given by $\vec{F}(\vec{s}(u,v)) = \langle u\sin(v), u\cos(v), 1-u^2 \rangle$

$\therefore \ \vec{F}(s(u,v)) \cdot (\vec{s}_u \times \vec{s}_v) = u(2u^2\sin(v)\cos(v) + 2u^2\sin(v)\cos(u) + 1-u^2)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{D_1} \vec{F}(\vec{r}_u \times \vec{r}_v)\, dA + \iint_{D_2} \vec{F}(\vec{S}_u \times \vec{S}_v)\, dA$$

$$= \iint_D 0\, dA + \iint_D u(4u^2\sin(v)\cos(v) + (-u^2))\, dA$$

$$= \int_0^1 \int_0^{2\pi} u(4u^2\sin(v)\cos(v) + 1-u^2)\, dv\, du$$

$$= \int_0^1 u\left[2u^2\sin^2(v) + v - vu^2\right]_{v=0}^{2\pi} du$$

$$= \int_{u=0}^1 u\,(0 + 2\pi - 0 - (2\pi - 0)u^2)\, du$$

$$= 2\pi \int_{u=0}^1 u(1-u^2)\, du = 2\pi\left(\frac{1}{2}u^2 - \frac{1}{4}u^4\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4} - 0\right)$$

$$= \boxed{\frac{\pi}{2}}$$

Stokes's Theorem

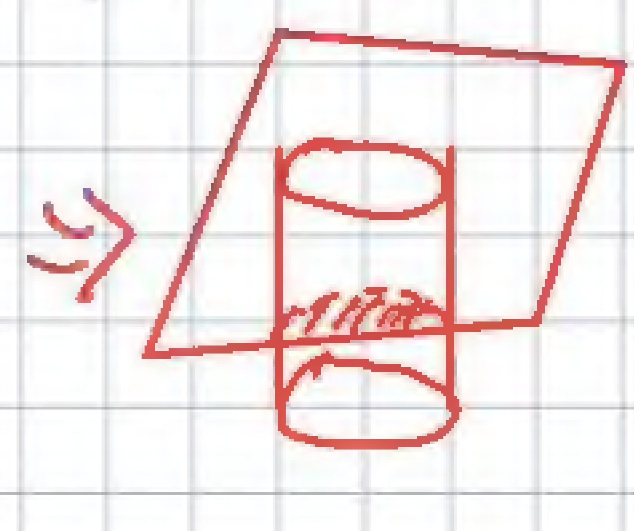
Idea. Generalize Green's Theorem to Surface are not flat

Prop: Suppose S is a piecewise smooth surface with piecewise smooth boundary curve which is closed and has only one component.

If a vector field with continuous partial derivatives on S,

$$\boxed{\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\, d\vec{r}}$$

Example: Compute $\int_C \vec{F}\, d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ and c the curve of intersections of plane $y+z=2$



Sol: $\vec{S}(r,\theta)=\langle r\cos(\theta), r\sin(\theta), 2-r\sin(\theta)\rangle$
on $(r,\theta)\in[0,1]\times[0,2\pi]$

By Stokes's Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_C \text{Curl}(\vec{F})\cdot d\vec{S}$$

$$= \iint_D \text{Curl}(\vec{F})(\vec{S}(r,\theta))\cdot(\vec{S}_u\times\vec{S}_v)\,dA$$

$$\text{Curl}(\vec{F}) = \nabla\times\vec{F} = \det\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z \end{vmatrix} = \langle 0,0,1+2y\rangle$$

$$\therefore \text{Curl}(\vec{F})(S(r,\theta)) = \langle 0,0,1+2r\sin(\theta)\rangle$$

$$\vec{S}_r = \langle\cos\theta, \sin\theta, -\sin\theta\rangle$$

$$\vec{S}_\theta = \langle -r\sin\theta, r\cos\theta, -r\cos\theta\rangle$$

$$\Rightarrow \vec{S}_r\times\vec{S}_\theta = \det\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -\sin\theta \\ -r\sin\theta & r\cos\theta & -r\cos\theta \end{vmatrix} = r\langle 0,1,1\rangle$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_D \langle 0,0,1+2r\sin(\theta)\rangle\cdot r\langle 0,1,1\rangle\,dA$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r(1+2r\sin\theta)\,d\theta\,dr$$

$$= \pi$$